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DERIVATION ALTERNATOR RINGS WITH IDEMPOTENT

BY

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ABSTRACT. A nonassociative ring is called a derivation alternator ring if it satisfies the identities $(yz, x, x) = y(z, x, x) + (y, x, x)z$, $(x, x, yz) = y(x, x, z) + (x, x, y)z$ and $(x, x, x) = 0$. Let R be a prime derivation alternator ring with idempotent $e \neq 1$ and characteristic $\neq 2$. If R is without nonzero nil ideals of index 2, then R is alternative.

1. Preliminaries. In this work we consider nonassociative rings with characteristic $\neq 2$ which satisfy the following identities:

$$(x, x, x) = 0, \tag{1}$$

$$(yz, x, x) = y(z, x, x) + (y, x, x)z, \tag{2}$$

$$(x, x, yz) = y(x, x, z) + (x, x, y)z, \tag{3}$$

where we employ the standard notaton $(x, y, z) = (xy)z - x(yz)$. It is immediate from (2), (3) and linearized (1) that such rings also satisfy

$$(x, yz, x) = y(x, z, x) + (x, y, x)z. \tag{4}$$

In fact, in conjunction with (1) any two of the three identities (2), (3) and (4) will imply the third. Since (2), (3) and (4) can be summarized by simply saying that the alternators of such rings are derivation maps, we shall henceforth refer to these rings as derivation alternator rings.

In [2] and [3] E. Kleinfeld defined two different generalizations of alternative rings, and for each of these generalizations he showed the simple rings with idempotent $e \neq 1$ to be alternative. Both of these generalizations defined by Kleinfeld are contained in the variety of derivation alternator rings, and in this work we extend his results to simple derivation alternator rings with idempotent $e \neq 1$ and characteristic $\neq 2$. In particular, our result eliminates the characteristic $\neq 3$ assumption used in [2].

In what follows we shall often employ the Teichmüller identity,

$$(wx, y, z) - (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y)z,$$

which holds in any ring. We shall also make use of the identity

$$[x \circ y, z] + [y \circ z, x] + [z \circ x, y] = 0, \tag{5}$$

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where $x \circ y = xy + yx$ and $[x, y] = xy - yx$. This last identity follows from linearized (1), since in any ring $[x \circ y, z] + [y \circ z, x] + [z \circ x, y] = (x, y, z) + (y, z, x) + (z, x, y) + (x, z, y) + (z, y, x) + (y, x, z)$. As in [2], Teichmüller together with (4), (3) and (1) gives $(x^2, y, x) = (x, xy, x) - (x, x, yx) + x(x, y, x) + (x, x, y)x = x(x, y, x) - (x, x, y)x + x(x, y, x) + (x, x, y)x = 2x(x, y, x)$. Thus we have shown

$$(x^2, y, x) = 2x(x, y, x). \tag{6}$$

It is to be noted, too, that when formed from a derivation alternator ring the standard opposite ring is likewise a derivation alternator ring. Thus by going to the opposite ring, (6) becomes

$$(x, y, x^2) = 2(x, y, x)x. \tag{7}$$

A nonassociative ring which satisfies the identity $(x, y, x) = 0$ is called flexible. The following proposition shows that in a flexible derivation alternator ring the additive span of the alternators forms an ideal. What we shall actually use later in the paper, however, is the identity (9) derived in the proof.

PROPOSITION. *In a flexible ring which satisfies identity* (3),

$$\begin{aligned} 2a(x, x, b) = &-\{(a, x^2, b) + (x^2, a, b)\} - \{(x^2, a, b) + (x^2, b, a)\} \\ &-\{(b, xa, x) + (xa, b, x)\} - \{(xb, a, x) + (a, xb, x)\} \\ &+\{(a, x, x \circ b) + (x, a, x \circ b)\} + (x, x, ab). \end{aligned} \tag{8}$$

PROOF. We first rewrite identity (3), its linearization and the Teichmüller identity in functional notation:

$$\begin{aligned} A(x, y, z) &= (x, x, yz) - y(x, x, z) - (x, x, y)z, \\ B(w, x, y, z) &= (w, x, yz) + (x, w, yz) - y\{(w, x, z) + (x, w, z)\} \\ &\quad - \{(w, x, y) + (x, w, y)\}z, \\ T(w, x, y, z) &= (wx, y, z) - (w, xy, z) + (w, x, yz) \\ &\quad - w(x, y, z) - (w, x, y)z. \end{aligned}$$

Then $0 = T(a, x, x, b) - T(b, x, a, x) - 2T(x, x, a, b) - 2T(x, b, a, x) - T(x, x, b, a) - T(x, a, b, x) + 3A(x, a, b) + A(x, b, a) + 2B(b, x, a, x) + B(a, x, b, x)$ implies after some cancellation and rearrangement of terms:

$$\begin{aligned} 0 = &\{(ax, x, b) + (b, x, ax)\} + 2\{(b, xa, x) + (x, xa, b)\} \\ &- \{(x \circ b, a, x) + (x, a, x \circ b)\} + \{(x, xb, a) + (a, xb, x)\} \\ &+ 2(x, ba, x) + (x, ab, x) \\ &- 2a\{(x, x, b) + (x, b, x) + (b, x, x)\} \\ &- b \circ \{(x, x, a) + (a, x, x)\} + 2x\{(x, a, b) + (b, a, x)\} \\ &+ x\{(x, b, a) + (a, b, x)\} - \{(b, x, a) + (a, x, b)\}x \\ &- \{(a, x^2, b) + (x^2, a, b)\} - \{(x^2, a, b) + (x^2, b, a)\} \\ &- \{(b, xa, x) + (xa, b, x)\} - \{(xb, a, x) + (a, xb, x)\} \\ &+ \{(a, x, x \circ b) + (x, a, x \circ b)\} + (x, x, ab) - 2a(x, x, b). \end{aligned} \tag{9}$$

If we now apply to (9) the flexible identity and its linearization, the first six lines reduce to zero, and so the last three give (8).

A nonassociative ring is called power-associative if each of its elements generates an associative subring. The following theorem shows derivation alternator rings have this property.

THEOREM 1. *A derivation alternator ring with characteristic $\neq 2$ is power-associative.*

PROOF. Let R be a derivation alternator ring with characteristic $\neq 2$ and $x \in R$. For $n > 1$ we define recursively $x^n = x^{n-1}x$ and show that $x^n = x^{n-i}x^i$ for $i = 1, 2, \ldots, n-1$. From (1), (2) and (3) we have $(x, x, x) = 0$ and $(x^2, x, x) = 0 = (x, x, x^2)$. Hence $x^n = x^{n-i}x^i$ holds for $n = 2, 3, 4$. We now induct on n, assuming $x^k = x^{k-i}x^i$ whenever $k < n$, and consider the case $n > 4$. To show $x^n = x^{n-i}x^i$ for $i = 1, 2, \ldots, n-1$, we make yet a second induction, this time on i. Now $x^n = x^{n-1}x$ holds by definition. Assume $x^n = x^{n-i}x^i$. From linearized (2) and our induction assumptions on n and i, we have

$$\begin{aligned} 2\left[x^n - x^{n-(i+1)}x^{i+1}\right] &= (x^{n-(i+1)}, x^i, x) + (x^{n-(i+1)}, x, x^i) \\ &= x^{n-(i+2)}\left[(x, x^i, x) + (x, x, x^i)\right] \\ &\quad + \left[(x^{n-(i+2)}, x^i, x) + (x^{n-(i+2)}, x, x^i)\right]x = 0. \end{aligned}$$

Thus $x^n = x^{n-(i+1)}x^{i+1}$ for $n - i \geqslant 3$. Next (3) and our induction assumption on n imply $(x, x, x^{n-2}) = x^{n-3}(x, x, x) + (x, x, x^{n-3})x = 0$, so that $x^2x^{n-2} = xx^{n-1}$. But (6) and our induction assumption on n imply $(x^2, x^{n-3}, x) = 2x(x, x^{n-3}, x) = 0$, whence also $x^{n-1}x = x^2x^{n-2}$. Thus $x^n = x^{n-1}x = x^2x^{n-2} = xx^{n-1}$, which completes both inductions and thereby the proof of the theorem.

2. Rings with idempotent. Henceforth we assume R to be a derivation alternator ring with idempotent $e \neq 1$ and characteristic $\neq 2$. Since by Theorem 1 such rings are power-associative, they permit the standard Albert decomposition **[1]** relative to e. Thus we have $R = R_1 + R_{1/2} + R_0$ (direct sum over the ring of integers) where $R_i = \{x \in R \mid e \circ x = 2ix\}$. Moreover, for $i = 0$ or 1 and $x_i, y_i \in R_i$ we have $ex_i = ix_i = x_ie$, $x_i \circ y_i \in R_i$ and $x_i \circ y_{1/2} \in R_{1/2} + R_{1-i}$. Also, $x_1y_0 = 0 = y_0x_1$ and $x_{1/2} \circ y_{1/2} \in R_1 + R_0$. In this section we shall show that for derivation alternator rings actually $x_iy_i \in R_i$ and $x_iy_{1/2}, y_{1/2}x_i \in R_{1/2}$ for $i = 0$ or 1.

LEMMA 1. $(e, R, e) = 0$ *and* $(R, e, e) = (e, e, R) \subseteq R_{1/2}$.

PROOF. Since always $(e, R_1, e) = 0 = (e, R_0, e)$, we need to show that also $(e, R_{1/2}, e) = 0$. Our proof is the same as in **[2]**. Let $x \in R_{1/2}$ and $ex = y_1 + y_{1/2} + y_0$ where $y_i \in R_i$. Then linearized (1) implies

$$\begin{aligned} (e, x, e) &= -(x, e, e) - (e, e, x) = -(xe)e + xe - ex + e(ex) \\ &= (ex - x)e + xe - ex + e(ex) = (ex) \circ e - ex \\ &= 2y_1 + y_{1/2} - y_1 - y_{1/2} - y_0 = y_1 - y_0. \end{aligned}$$

But by (6), $(e, x, e) = (e^2, x, e) = 2e(e, x, e)$, so that $y_1 - y_0 = 2e(y_1 - y_0) = 2y_1$.

Hence $y_1 = 0 = y_0$ and $(e, R_{1/2}, e) = 0$. Since we now have $(e, R, e) = 0$, linearized (1) implies $(R, e, e) = (e, e, R)$. Also, (3) and (1) imply $(e, e, x) = (e, e, e \circ x) = e \circ (e, e, x) \in R_{1/2}$. Hence $(e, e, R) \subseteq R_{1/2}$.

THEOREM 2. *Let R be a derivation alternator ring with idempotent e and characteristic* $\neq 2$. *Then* $R = R_1 + R_{1/2} + R_0$ *where* $R_i = \{x \in R \mid e \circ x = 2ix\}$. *Furthermore,* $R_1R_0 = 0 = R_0R_1$, $R_i^2 \subseteq R_i$, $R_iR_{1/2} \subseteq R_{1/2}$ *and* $R_{1/2}R_i \subseteq R_{1/2}$ *for* $i = 0, 1$.

PROOF. Throughout we set $x_i, y_i \in R_i$ for $i = 0, 1$. From (2) we have $(x_iy_i, e, e) = x_i(y_i, e, e) + (x_i, e, e)y_i = 0$, whence Teichmüller gives

$$(x_i, y_i, e)e = (x_iy_i, e, e) - (x_i, y_ie, e) + (x_i, y_i, e^2) - x_i(y_i, e, e)$$
$$= (1 - i)(x_i, y_i, e).$$

Then using linearized (2) we have

$$(1 - i)(x_i, y_i, e) = (x_i, y_i, e)e = [(x_i, y_i, e) + (x_i, e, y_i)]e$$
$$= [(x_ie, y_i, e) + (x_ie, e, y_i)] - x_i[(e, y_i, e) + (e, e, y_i)]$$
$$= i[(x_i, y_i, e) + (x_i, e, y_i)] = i(x_i, y_i, e),$$

so that $(x_i, y_i, e) = 0$. Thus $(x_iy_i)e = i(x_iy_i)$. Going to the opposite ring now gives that also $e(x_iy_i) = i(x_iy_i)$, and therefore $R_i^2 \subseteq R_i$.

Next, to prove $R_iR_{1/2} \subseteq R_{1/2}$, we let $x \in R_{1/2}$. Then from Teichmüller and (3) we obtain

$$(x_ie, e, x) - (x_i, e^2, x) + (x_i, e, ex) = x_i(e, e, x) + (x_i, e, e)x = x_i(e, e, x)$$
$$= (e, e, x_ix) - (e, e, x_i)x = (e, e, x_ix).$$

Thus $(i - 1)(x_i, e, x) + (x_i, e, ex) = (e, e, x_ix) \in R_{1/2}$ by Lemma 1. Going to the opposite ring, this in turn gives $(i - 1)(x, e, x_i) + (xe, e, x_i) \in R_{1/2}$. Since $(z, e, z) = e \circ (z, e, z) \in R_{1/2}$ by (4), it then follows that also $(i - 1)(x_i, e, x) + (x_i, e, xe) \in R_{1/2}$. Thus we have shown $(2i - 1)(x_i, e, x) = \{(i - 1)(x_i, e, x) + (x_i, e, ex)\} + \{(i - 1)(x_i, e, x) + (x_i, e, xe)\} \in R_{1/2}$, so that

(i) $(x_i, e, x) \in R_{1/2}$.

From linearized (3) we next obtain

$$(x_i, e, x) + (e, x_i, x) = (x_i, e, e \circ x) + (e, x_i, e \circ x)$$
$$= e \circ [(x_i, e, x) + (e, x_i, x)] + x \circ [(x_i, e, e) + (e, x_i, e)]$$
$$= e \circ [(x_i, e, x) + (e, x_i, x)].$$

Thus $(x_i, e, x) + (e, x_i, x) \in R_{1/2}$, so that applying (i) we have

(ii) $(e, x_i, x) \in R_{1/2}$.

Since $(e, z, z) = e \circ (e, z, z) \in R_{1/2}$ by (2), this in turn gives $(e, x, x_i) \in R_{1/2}$. Then going to the opposite ring we have

(iii) $(x_i, x, e) \in R_{1/2}$.

Finally, $x_ix - e \circ (x_ix) = -(x_i, e, x) + (e, x_i, x) - (x_i, x, e) \in R_{1/2}$ by (i), (ii) and (iii). Setting $x_ix = y_1 + y_{1/2} + y_0$, this implies $-y_1 + y_0 \in R_{1/2}$ or that $y_1 = y_0 = 0$. Thus we have established $R_iR_{1/2} \subseteq R_{1/2}$. Since going to the opposite ring now gives $R_{1/2}R_i \subseteq R_{1/2}$ as well, this completes the proof of the theorem.

3. Semiprime rings. We define the flexible nucleus of a nonassociative ring R to be

$$N_F(R) = \{r \in R | 0 = (x, r, x) = (r, x, r) = (r, x, y) + (y, x, r) \quad \text{for all } x, y \in R\}.$$

In this section we shall show that in a semiprime derivation alternator ring with characteristic $\neq 2$ every idempotent must be in this nucleus.

LEMMA 2. *$Z = \{z \in R_{1/2} | [z, R] = 0 = zR_{1/2}\}$ is a trivial ideal of R.*

PROOF. Since $[Z, R] = 0$ and $R_{1/2}Z = 0 \subseteq Z$, it suffices to show that if $z \in Z$ and $x_i \in R_i$, where $i = 0$ or 1, then $x_i z \in Z$. We note that $x_i z \in R_{1/2}$ by Theorem 2. Now $0 = [e \circ x_i, z] + [x_i \circ z, e] + [e \circ z, x_i] = [x_i \circ z, e] + [z, x_i] = 2[x_i z, e]$ by (5). Since this implies $[x_i z, e] = 0$, and since $x_i z \in R_{1/2}$, it follows that $x_i z = 2e(x_i z)$. Also, $[e, z] = 0$ and $z \in R_{1/2}$ imply $z = 2ez$. Now from (3) and the definition of Z we have $[(w, w, z), y] = [(w, w, z), y] + [z, (w, w, y)] = (w, w, [z, y]) = 0$. In particular, this implies

$$0 = 2\big[(e, x_i, z) + (x_i, e, z), y\big] = (4i - 2)\big[x_i z, y\big].$$

Thus we have shown $[RZ, R] = 0$.

Next let $y \in R_{1/2}$. Since $ZR_{1/2} = 0$ and by Theorem 2, $R_{1/2}R_i \subseteq R_{1/2}$ for $i = 0$ or 1, we have $(z, y, x_i) = 0$. Hence from linearized (2) we obtain

$$\begin{aligned}(z, x_i, y) &= (e \circ z, x_i, y) + (e \circ z, y, x_i)\\ &= e \circ \big[(z, x_i, y) + (z, y, x_i)\big] + z \circ \big[(e, x_i, y) + (e, y, x_i)\big]\\ &= e \circ (z, x_i, y).\end{aligned}$$

This means $(x_i z)y = (z, x_i, y) = e \circ (z, x_i, y) = e \circ ((x_i z)y) \in R_{1/2}$. But then $[RZ, R] = 0$ and $x \circ y \in R_1 + R_0$ for $x, y \in R_{1/2}$ imply $2(x_i z)y = (x_i z) \circ y \in R_{1/2} \cap (R_1 + R_0) = 0$. Thus we have shown $(RZ)R_{1/2} = 0$, which completes the proof that Z is an ideal of R. Moreover, Z is trivial since $Z^2 \subseteq ZR_{1/2} = 0$.

LEMMA 3. *Let $H = \{h \in R_{1/2} | [e, h] = 0\}$. Then $HR_i \subseteq H$ and $R_i H \subseteq H$ for $i = 0, 1$.*

PROOF. Let $h \in H$ and $x_i \in R_i$ for $i = 0, 1$. From (2) and the definition of H we have $0 = ([e, h], w, w) = [(e, w, w), h] + [e, (h, w, w)]$ for all $w \in R$. In particular, $0 = [(e, e, x_i) + (e, x_i, e), h] + [e, (h, e, x_i) + (h, x_i, e)] = [e, (h, e, x_i) + (h, x_i, e)]$. Since $(h, e, x_i) + (h, x_i, e) \in R_{1/2}$ by Theorem 2, it thus follows $(h, e, x_i) + (h, x_i, e) \in H$. Now $h = 2he$, so expansion of $2[(h, e, x_i) + (h, x_i, e)]$ gives $(1 - 4i)hx_i + 2(hx_i)e \in H$. Hence we obtain

(iv) $(4i - 1)hx_i \equiv 2(hx_i)e \bmod H$.

Then, since $0 = [e, h] = 2[e, he]$ implies $He \subseteq H$, multiplying (iv) through on the right by $2e$ gives $(8i - 2)(hx_i)e \equiv 4[(hx_i)e]e \bmod H$. But using (2) and $4(h, e, e) = 2he - 4he = -h$, we have $4[(hx_i)e]e = 4(hx_i, e, e) + 4(hx_i)e = 4[h(x_i, e, e) + (h, e, e)x_i] + 4(hx_i)e = -hx_i + 4(hx_i)e$. Thus we arrive at $(8i - 2)(hx_i)e \equiv -hx_i + 4(hx_i)e \bmod H$, or

(v) $hx_i \equiv (3 - 4i)2(hx_i)e \bmod H$.

Finally, combining (iv) and (v) leads mod H to $hx_i \equiv (3 - 4i)2(hx_i)e \equiv (3 - 4i)(4i - 1)hx_i = -3hx_i$ for $i = 0$ or 1. But then $4hx_i \equiv 0 \bmod H$, so that $HR_i \subseteq H$. Going to the opposite ring now gives $R_iH \subseteq H$ as well.

LEMMA 4. *$(e, x, y_{1/2}) + (y_{1/2}, x, e) \in Z$ for $y_{1/2} \in R_{1/2}$ and $x \in R$.*

PROOF. First linearization of (6) and (7) together with Lemma 1 gives $(e^2, x, y_{1/2}) + (e \circ y_{1/2}, x, e) = 2e[(e, x, y_{1/2}) + (y_{1/2}, x, e)]$ and $(y_{1/2}, x, e^2) + (e, x, e \circ y_{1/2}) = 2[(e, x, y_{1/2}) + (y_{1/2}, x, e)]e$. Adding these two equations and dividing by 2, we find that $(e, x, y_{1/2}) + (y_{1/2}, x, e) = e \circ [(e, x, y_{1/2}) + (y_{1/2}, x, e)] \in R_{1/2}$. Also, subtracting one equation from the other, we obtain $[e, (e, x, y_{1/2}) + (y_{1/2}, x, e)] = 0$. In particular, this shows

(vi) $(e, x, y_{1/2}) + (y_{1/2}, x, e) \in H$.

Next let $x_{1/2} \in R_{1/2}$. Then using Theorem 2 and the fact $x_{1/2} \circ y_{1/2} \in R_1 + R_0$ in the Albert decomposition, we have

$$\begin{aligned}(e, x_{1/2}, y_{1/2}) &+ (y_{1/2}, x_{1/2}, e)\\ &= (ex_{1/2})y_{1/2} - e(x_{1/2}y_{1/2}) + (y_{1/2}x_{1/2})e - y_{1/2}(x_{1/2}e)\\ &= (ex_{1/2}) \circ y_{1/2} - e(x_{1/2} \circ y_{1/2})\\ &\quad + (y_{1/2}x_{1/2}) \circ e - y_{1/2}x_{1/2} \in R_1 + R_0.\end{aligned}$$

But this together with (vi) implies

(vii) $(e, x_{1/2}, y_{1/2}) + (y_{1/2}, x_{1/2}, e) \in (R_1 + R_0) \cap R_{1/2} = 0$.

Let $x_i \in R_i$ for $i = 0, 1$. Then from linearized (4), Theorem 2 and (vii) we have $[(e, x_i, y_{1/2}) + (y_{1/2}, x_i, e)]x_{1/2} = [(e, x_ix_{1/2}, y_{1/2}) + (y_{1/2}, x_ix_{1/2}, e)] - x_i[(e, x_{1/2}, y_{1/2}) + (y_{1/2}, x_{1/2}, e)] = 0$. Thus in conjunction with (vii) we have shown $[(e, x, y_{1/2}) + (y_{1/2}, x, e)]R_{1/2} = 0$. If we now also go to the opposite ring, it follows that

(viii) $R_{1/2}[(e, x, y_{1/2}) + (y_{1/2}, x, e)] = 0 = [(e, x, y_{1/2}) + (y_{1/2}, x, e)]R_{1/2}$.

Lastly, let $h = (e, x, y_{1/2}) + (y_{1/2}, x, e)$ and again $x_i \in R_i$ for $i = 0, 1$. Then $0 = [e \circ x_i, h] + [x_i \circ h, e] + [h \circ e, x_i] = (2i - 1)[x_i, h] + [x_i \circ h, e]$ by (5). But $[x_i \circ h, e] = 0$ by (vi) and Lemma 3, so that $(2i - 1)[x_i, h] = 0$. Hence we obtain

(ix) $[R_i, (e, x, y_{1/2}) + (y_{1/2}, x, e)] = 0$ for $i = 0, 1$.

The lemma now follows from (vi), (viii) and (ix).

THEOREM 3. *If R is a semiprime derivation alternator ring with idempotent e and characteristic $\neq 2$, then $e \in N_F(R)$.*

PROOF. It will suffice to show $(e, x, y) + (y, x, e) = 0$ for all $x, y \in R$, since then also $(x, e, x) = -\{(e, x, x) + (x, x, e)\} = 0$ by linearized (1). Now since R is semiprime, from Lemmas 2 and 4 we have

(x) $(e, x, y_{1/2}) + (y_{1/2}, x, e) = 0$ for $x \in R$ and $y_{1/2} \in R_{1/2}$.

Also, from Theorem 2 we have

(xi) $(e, x_j, y_i) = 0 = (y_i, x_j, e)$ for $x_j \in R_j$ and $y_i \in R_i$ where $i, j = 0, 1$.

We next consider $x \in R_{1/2}$ and $y_i \in R_i$ for $i = 0, 1$. From linearization of (6) and

(7) together with Lemma 1 we obtain $(e^2, x, y_i) + (e \circ y_i, x, e) = 2e[(e, x, y_i) + (y_i, x, e)]$ and $(y_i, x, e^2) + (e, x, e \circ y_i) = 2[(e, x, y_i) + (y_i, x, e)]e$. Since $w = (e, x, y_i) + (y_i, x, e) \in R_{1/2}$ by Theorem 2, adding these last two equatons gives $(2i + 1)w = 2e \circ w = 2w$. Hence $(2i - 1)w = 0$, so that

(xii) $(e, x_{1/2}, y_i) + (y_i, x_{1/2}, e) = 0$ for $x_{1/2} \in R_{1/2}$ and $y_i \in R_i$ where $i = 0, 1$.

Since (x), (xi) and (xii) show that in fact $(e, x, y) + (y, x, e) = 0$ for all $x, y \in R$, this completes the proof of the theorem.

4. A nil ideal. In this section we shall assume the conclusion of Theorem 3, namely that $e \in N_F(R)$, and consider the set

$$B = \{b \in R_{1/2} | bR_{1/2} \subseteq R_{1/2} \text{ and } R_{1/2}b \subseteq R_{1/2}\}.$$

In [2], B proved to be a nil ideal. It turns out that this result carries over to derivation alternator rings.

LEMMA 5. $(R, e, e) = (e, e, R) \subseteq B$.

PROOF. From Lemma 1 we already know $(R, e, e) = (e, e, R) \subseteq R_{1/2}$. Let $a, b \in R_{1/2}$. Then applying $e \in N_F(R)$ to (9) with $x = e$ we obtain (8):

$$\begin{aligned} 2a(e, e, b) = &-\{(a, e^2, b) + (e^2, a, b)\} - \{(e^2, a, b) + (e^2, b, a)\} \\ &-\{(b, ea, e) + (ea, b, e)\} - \{(eb, a, e) + (a, eb, a)\} \\ &+\{(a, e, e \circ b) + (e, a, e \circ b)\} + (e, e, ab), \end{aligned}$$

or

$$\begin{aligned} 2a(e, e, b) = &\{(b, a, e) + (a, b, e)\} - \{(b, ea, e) + (ea, b, e)\} \\ &-\{(eb, a, e) + (a, eb, e)\} + (e, e, ab). \end{aligned}$$

But $(x, x, e) = (x, x, e^2) = e \circ (x, x, e) \in R_{1/2}$ by (3), so that this last equation implies $R_{1/2}(e, e, R) \subseteq R_{1/2}$. Since by going to the opposite ring this in turn gives $(R, e, e)R_{1/2} \subseteq R_{1/2}$, we have established $(R, e, e) = (e, e, R) \subseteq B$.

LEMMA 6. $(ex_{1/2}) \circ (ey_{1/2}) = 0 = (x_{1/2}e) \circ (y_{1/2}e)$ *for* $x_{1/2}, y_{1/2} \in R_{1/2}$.

PROOF. Let $x, y \in R_{1/2}$. We note that $xe \in R_{1/2}$ by Theorem 2, so that $(x, e, e) = (xe)e - xe = -e(xe)$. Thus using $e \in N_F(R)$ and Teichmüller we obtain

$$\begin{aligned} (ex) \circ (ey) + (x, e, e)y + (y, e, e)x &= (ex) \circ (ey) - [(ex)e]y - [(ey)e]x \\ &= -(ex, e, y) - (ey, e, x) = -(ex, e, y) + (x, e, ey) \\ &= (xe, e, y) - (x, e, y) + (x, e, ey) \\ &= x(e, e, y) + (x, e, e)y. \end{aligned}$$

This shows $(ex) \circ (ey) = x(e, e, y) - (y, e, e)x = x \circ (e, e, y)$, again using $e \in N_F(R)$. But $(e, e, y) \in B$ from Lemma 5, so that $x \circ (e, e, y) \in (R_1 + R_0) \cap R_{1/2} = 0$. Hence $(ex) \circ (ey) = 0$, and going to the opposite ring $(xe) \circ (ye) = 0$ as well.

LEMMA 7. $(e, R, R) + (R, e, R) + (R, R, e) \subseteq R_{1/2}$.

PROOF. By Theorem 2 it suffices to show $(e, R_{1/2}, R_{1/2}), (R_{1/2}, e, R_{1/2})$ and $(R_{1/2}, R_{1/2}, e)$ are in $R_{1/2}$. Now in any ring $[x, y \circ z] - y \circ [x, z] - z \circ [x, y] = \{(y, x, z) + (z, x, y)\} - \{(x, y, z) + (z, y, x)\} - \{(y, z, x) + (x, z, y)\}$. Thus from $e \in N_F(R)$ we have

(xiii) $[x, y \circ z] = y \circ [x, z] + z \circ [x, y]$ if any of x, y, z equals e.

Let $x, y \in R_{1/2}$. Using Theorem 2 and (xiii) we obtain $[ex, ey] = [ex, e \circ (ey)] = e \circ [ex, ey] + (ey) \circ [ex, e]$. But $(ey) \circ [ex, e] = (ey) \circ [(ex)e - e(ex)] = (ey) \circ [e(xe)] - (ey) \circ [e(ex)] = 0$ by Lemma 1, Theorem 2 and Lemma 6. Thus $[ex, ey] = e \circ [ex, ey] \in R_{1/2}$. Now using Lemma 6 we have $2(ex)(ey) = [ex, ey] \in R_{1/2}$, so that $(ex)(ey) \in R_{1/2}$. Since $(ex, e, y) = [(ex)e]y - (ex)(ey) = -(x, e, e)y - (ex)(ey)$, this and Lemma 5 imply $(ex, e, y) \in R_{1/2}$. The assumption $e \in N_F(R)$ then shows $(y, e, ex) \in R_{1/2}$, whence going to the opposite ring gives also $(xe, e, y) \in R_{1/2}$. Thus it follows $(x, e, y) = (ex, e, y) + (xe, e, y) \in R_{1/2}$.

Next using linearized (2) we have $(x, y, e) + (x, e, y) = (e \circ x, y, e) + (e \circ x, e, y) = e \circ [(x, y, e) + (x, e, y)] + x \circ [(e, y, e) + (e, e, y)] = e \circ [(x, y, e) + (x, e, y)]$ by Lemmas 1 and 5. Since we have already established $(x, e, y) = e \circ (x, e, y) \in R_{1/2}$, this shows $(x, y, e) = e \circ (x, y, e) \in R_{1/2}$ as well. Finally, going to the opposite ring now gives $(e, y, x) \in R_{1/2}$, which completes the proof of the lemma.

LEMMA 8. *$(R_{1/2}, R_{1/2}, R_i) + (R_i, R_{1/2}, R_{1/2}) \subseteq R_{1/2}$ for $i = 0, 1$.*

PROOF. First let $y_i \in R_i$ for $i = 0, \frac{1}{2}, 1$. Then (4) and $e \in N_F(R)$ imply $(x, e \circ y_i, x) = e \circ (x, y_i, x) + y_i \circ (x, e, x) = e \circ (x, y_i, x)$, whence it follows that

(xiv) $(x, R_i, x) \subseteq R_i$ for $i = 0, \frac{1}{2}, 1$.

Now let $x, z \in R_{1/2}$ and $y_i \in R_i$ for $i = 0, 1$. From Teichmüller, Lemma 7 and Theorem 2, we have $(ex, z, y_i) - e(x, z, y_i) = (e, xz, y_i) - (e, x, zy_i) + (e, x, z)y_i \in R_{1/2}$. Then going to the opposite ring gives $(y_i, z, xe) - (y_i, z, x)e \in R_{1/2}$. Since $z \in R_{1/2}$, this last containment together with (xiv) implies $(xe, z, y_i) - (x, z, y_i)e \in R_{1/2}$. Hence, now $(x, z, y_i) - e \circ (x, z, y_i) = [(ex, z, y_i) - e(x, z, y_i)] + [(xe, z, y_i) - (x, z, y_i)e] \in R_{1/2}$, so that actually $(x, z, y_i) \in R_{1/2}$. Since going to the opposite ring also implies that $(y_i, z, x) \in R_{1/2}$, this completes the proof of the lemma.

THEOREM 4. *If R is a derivaton alternator ring with idempotent $e \in N_F(R)$ and characteristic $\neq 2$, then $B = \{b \in R_{1/2} | bR_{1/2} \subseteq R_{1/2}$ and $R_{1/2}b \subseteq R_{1/2}\}$ is a nil ideal of index 2.*

PROOF. As indicated in §2, in the Albert decomposition $s \circ t \in R_1 + R_0$ when $s, t \in R_{1/2}$. Thus we first note that if $b \in B$ and $x \in R_{1/2}$, then $b \circ x \in (R_1 + R_0) \cap R_{1/2} = 0$. In particular, $b^2 = 0$ for each $b \in B$.

From Theorem 2 and the definition of B, it is clear that both $RB \subseteq R_{1/2}$ and $BR \subseteq R_{1/2}$. Let $b \in B$, $x \in R_{1/2}$ and $y_i \in R_i$ for $i = 0, 1$. By Lemma 8 and Theorem 2, $x(by_i) = -(x, b, y_i) + (xb)y_i \in R_{1/2}$, so by going to the opposite ring also $(y_ib)x \in R_{1/2}$. Now (3) implies $(z, z, e) = e \circ (z, z, e) \in R_{1/2}$ for any z. Hence $(z, z, b) = (z, z, e \circ b) = e \circ (z, z, b) + b \circ (z, z, e) = e \circ (z, z, b) \in R_{1/2}$ by (3) and our initial note. But then $(x, y_i, b) + (y_i, x, b) \in R_{1/2}$, so that $(x, y_i, b) \in R_{1/2}$ using Theorem 2 and the definition of B. This and Theorem 2 imply $x(y_ib) = -(x, y_i, b) + (xy_i)b \in R_{1/2}$. Since going to the opposite ring shows also $(by_i)x \in R_{1/2}$, we have now established $R_iB \subseteq B$ and $BR_i \subseteq B$ for $i = 0, 1$.

Next let $b \in B$ and $x, y \in R_{1/2}$. Since $b \circ y = 0$, we have $(by) \circ x = (b, y, x) + (x, y, b) + b(yx) - (xy)b \in R_{1/2}$, using (xiv) from the proof of Lemma 8 and the definition of B. Thus in fact $(by) \circ x \in (R_1 + R_0) \cap R_{1/2} = 0$. If we now apply (xiii) from the proof of Lemma 7, we obtain $[x, by] = [x, e \circ (by)] = e \circ [x, by] + (by) \circ [x, e] = e \circ [x, by] \in R_{1/2}$, since $(by) \circ [x, e] \in (by) \circ R_{1/2} = 0$. Consequently $2x(by) = [x, by] + (by) \circ x = [x, by] \in R_{1/2}$, so that $x(by) = -x(yb) = -(by)x = (yb)x \in R_{1/2}$. Thus we have shown $R_{1/2}B \subseteq B$ and $BR_{1/2} \subseteq B$, which completes the proof B is an ideal of R.

5. Rings without nil ideals. In §3 we deduced from $Z = 0$ that e is in the flexible nucleus $N_F(R)$. In this section we deduce similarly from $B = 0$ that e is in the alternative nucleus $N_A(R)$. This latter nucleus was defined by M. Rich [4] as

$$N_A(R) = \{r \in R \mid (x, r, x) = 0 \text{ and } (r, y, x) = (y, x, r) = (x, r, y) \text{ for all } x, y \in R\}.$$

For the time being we assume that R is a derivation alternator ring with characteristic $\neq 2$, that e is an idempotent in $N_F(R)$ and that the ideal B of §4 is zero. Then Lemma 5 gives $(R, e, e) = 0 = (e, e, R)$, and from Lemma 1 we also have $(e, R, e) = 0$. Thus the Albert decomposition of R relative to e can now be refined to the Peirce decomposition $R = R_{11} + R_{10} + R_{01} + R_{00}$, where $R_{ij} = \{x \in R \mid ex = ix$ and $xe = jx\}$ for $i, j = 0, 1$.

LEMMA 9.

	R_{11}	R_{10}	R_{01}	R_{00}
R_{11}	R_{11}	R_{10}	0	0
R_{10}	0	$R_{10} + R_{01}$	$R_{11} + R_{10} + R_{01}$	R_{10}
R_{01}	R_{01}	$R_{10} + R_{01} + R_{00}$	$R_{10} + R_{01}$	0
R_{00}	0	0	R_{01}	R_{00}

PROOF. First, $e \in N_F(R)$ gives $0 = (x_{ii}, e, x_{ij}) + (x_{ij}, e, x_{ii}) = (j - i)x_{ij}x_{ii}$ for $i \neq j$, so that $R_{ij}R_{ii} = 0$. Then going to the opposite ring we also have $R_{ii}R_{ji} = 0$. Now again using $e \in N_F(R)$, $0 = (x_{ji}, x_{ii}, e) + (e, x_{ii}, x_{ji}) = (x_{ji}, x_{ii}, e) = (x_{ji}x_{ii})e - ix_{ji}x_{ii}$ for $i \neq j$. Thus $x_{ji}x_{ii} \in R_{ji} + R_{ii}$. But also $x_{ji}x_{ii} \in R_{ij} + R_{ji}$ by Theorem 2, and together these two containments imply $R_{ji}R_{ii} \subseteq R_{ji}$. Then going to the opposite ring we also have $R_{ii}R_{ij} \subseteq R_{ij}$.

Next, by Lemma 7, $(R_{ij}, R_{ji}, e) \subseteq R_{ij} + R_{ji}$ for $i \neq j$, which implies $R_{ij}R_{ji} \subseteq R_{ii} + R_{ij} + R_{ji}$. Then $e \in N_F(R)$ gives $0 = (e, x_{ij}, y_{ij}) + (y_{ij}, x_{ij}, e) = ix_{ij}y_{ij} - e(x_{ij}y_{ij}) + (y_{ij}x_{ij})e - jy_{ij}x_{ij}$. Since for $i \neq j$ by Lemma 6, $x_{ij} \circ y_{ij} = 0$, this last equation implies $x_{ij}y_{ij} = (i + j)x_{ij}y_{ij} = e \circ (x_{ij}y_{ij})$. Thus $R_{ij}^2 \subseteq R_{ij} + R_{ji}$.

Finally, noting that from Theorem 2, $R_{ii}^2 \subseteq R_{ii}$ and $R_{ii}R_{jj} = 0$ for $i \neq j$, the proof of the lemma is complete.

LEMMA 10. *If* $x \in R_{1/2}$, *then* $(x, R, x) = 0$.

PROOF. Linearized (6), $e \in N_F(R)$ and Lemma 1 imply $(e^2, y, z) + (e \circ z, y, e) = 2e[(e, y, z) + (z, y, e)] + 2z(e, y, e) = 0$. In particular, if $z_i \in R_i$ where $i = 0$ or 1, then $0 = (e, y, z_i) + 2i(z_i, y, e) = (2i - 1)(z_i, y, e)$ implies $(z_i, y, e) = 0$. Let $x \in R_{1/2}$. Then in the Albert decomposition $x^2 \in R_1 + R_0$, so that linearized (6) and $e \in N_F(R)$ imply $(x, y, x) = (e \circ x, y, x) + (x^2, y, e) = 2x[(x, y, e) + (e, y, x)] + 2e(x, y, x) = 2e(x, y, x)$. Thus $(x, y, x) = 4(e, e, (x, y, x)) = 0$ by Lemma 5 and our assumption $B = 0$.

LEMMA 11. *If $x \in R_{1/2}$, then $(x, x, e) = 0$.*

PROOF. Let $x \in R_{1/2}$. To show $(x, x, e) = 0$, we shall show $(x, x, e) \in B$. First, $(x, x, e) \in R_{1/2}$ by Lemma 7. Now let $a \in R_{1/2}$ and set $b = e$ in (9). Then all the associators in (9) which disappear or cancel when R is flexible likewise do so here. For if e appears this follows from $e \in N_F(R)$, and otherwise it follows from Lemma 10 and $x, a \in R_{1/2}$. We are thus left with (8):

$$\begin{aligned} 2a(x, x, e) = &-\{(a, x^2, e) + (x^2, a, e)\} - \{(x^2, a, e) + (x^2, e, a)\} \\ &-\{(e, xa, x) + (xa, e, x)\} - \{(xe, a, x) + (a, xe, x)\} \\ &+\{(a, x, x) + (x, a, x)\} + (x, x, ae). \end{aligned}$$

Then applying Lemmas 7 and 10 to this last equation we have

(xv) $2a(x, x, e) \equiv -(xe, a, x) - (a, xe, x) + (a, x, x) + (x, x, ae) \bmod R_{1/2}$.

We next let $x = x_{10} + x_{01}$ and consider two cases, namely $a \in R_{10}$ and $a \in R_{01}$. First, from (xv) we have

$$2a_{10}(x, x, e) \equiv -(x_{01}, a_{10}, x) - (a_{10}, x_{01}, x) + (a_{10}, x, x) \bmod R_{1/2}.$$

Now using Lemma 10, $(x_{01}, a_{10}, x) = (x_{01}, a_{10}, x_{10})$ and $-(a_{10}, x_{01}, x) + (a_{10}, x, x) = (a_{10}, x_{10}, x) = (a_{10}, x_{10}, x_{10}) + (a_{10}, x_{10}, x_{01}) = (a_{10}, x_{10}, x_{10}) - (x_{01}, x_{10}, a_{10})$. Thus we arrive at

$$2a_{10}(x, x, e) \equiv -(x_{01}, a_{10}, x_{10}) + (a_{10}, x_{10}, x_{10}) - (x_{01}, x_{10}, a_{10}) \bmod R_{1/2}.$$

Let $w = -(x_{01}, a_{10}, x_{10}) + (a_{10}, x_{10}, x_{10}) - (x_{01}, x_{10}, a_{10})$. By Lemma 6, $x_{ij} \circ a_{ij} = 0$ for $i \neq j$, and in particular $x_{ij}^2 = 0$. Hence $w = -(x_{01}a_{10})x_{10} + (a_{10}x_{10})x_{10} - (x_{01}x_{10})a_{10} \in RR_{10}$, which is contained in $R_{1/2} + R_0$ by Lemma 9. But from Lemmas 10, 6 and 9 we also have $w = (x_{10}, a_{10}, x_{01}) - (x_{10}, x_{10}, a_{10}) + (a_{10}, x_{10}, x_{01}) = -x_{10}(a_{10}x_{01}) + x_{10}(x_{10}a_{10}) - a_{10}(x_{10}x_{01}) \in R_1 + R_{1/2}$. Together these two containments imply $w \in R_{1/2}$, whence it follows that $a_{10}(x, x, e) \equiv 0 \bmod R_{1/2}$.

In the second case, from (xv) we have:

$$2a_{01}(x, x, e) \equiv -(x_{01}, a_{01}, x) - (a_{01}, x_{01}, x) + (a_{01}, x, x) + (x, x, a_{01}) \bmod R_{1/2}.$$

Now by Lemma 10, $(x_{01}, a_{01}, x) = (x_{01}, a_{01}, x_{10})$, $(a_{01}, x_{01}, x) = (a_{01}, x_{01}, x_{10}) + (a_{01}, x_{01}, x_{01}) = (a_{01}, x_{01}, x_{10}) - (x_{01}, x_{01}, a_{01})$ and $(a_{01}, x, x) + (x, x, a_{01}) = 0$. Thus we arrive at

$$2a_{01}(x, x, e) \equiv -(x_{01}, a_{01}, x_{10}) - (a_{01}, x_{01}, x_{10}) + (x_{01}, x_{01}, a_{01}) \bmod R_{1/2}.$$

Let $v = -(x_{01}, a_{01}, x_{10}) - (a_{01}, x_{01}, x_{10}) + (x_{01}, x_{01}, a_{01})$. Then, like for w above, Lemmas 6 and 9 give $v = x_{01}(a_{01}x_{10}) + a_{01}(x_{01}x_{10}) - x_{01}(x_{01}a_{01}) \in R_{1/2} + R_0$. But

Lemmas 10, 6 and 9 also show that $v = (x_{10}, a_{01}, x_{01}) + (x_{10}, x_{01}, a_{01}) - (a_{01}, x_{01}, x_{01}) = (x_{10}a_{01})x_{01} + (x_{10}x_{01})a_{01} - (a_{01}x_{01})x_{01} \in R_1 + R_{1/2}$. Since together these two containments imply $v \in R_{1/2}$, we thus have $a_{01}(x, x, e) \equiv 0 \bmod R_{1/2}$.

The preceding two cases establish $a(x, x, e) \in R_{1/2}$. Going to the opposite ring this in turn gives $(e, x, x)a \in R_{1/2}$, whence $e \in N_F(R)$ implies $(x, x, e)a \in R_{1/2}$. Hence $(x, x, e) \in B = 0$, which completes the proof of the lemma.

THEOREM 5. *Let R be a derivation alternator ring with idempotent e and characteristic $\neq 2$. If R has no nonzero nil ideal of index 2 contained in $R_{1/2}$, then $e \in N_A(R)$.*

PROOF. From Lemma 2, Z is a nil ideal of index 2 contained in $R_{1/2}$. Thus under our present assumptions $Z = 0$, and so the proof of Theorem 3 shows $e \in N_F(R)$. But then, by Theorem 4, B too is a nil ideal of index 2 contained in $R_{1/2}$, which likewise must be zero. Hence the assumptions we made at the start of this section are now all realized.

Our goal is to show $e \in N_A(R)$. Since already $e \in N_F(R)$, it only remains to establish $(x, y, e) + (x, e, y) = 0$. Moreover, it may be verified by use of Lemma 9 that to show $(x, y, e) + (x, e, y) = 0$ it suffices to show $(x_{ij}, y, e) + (x_{ij}, e, y) = 0$ for $y \in R_{ij}$ or R_{ji} and $i \neq j$.

First we have from Lemma 11 that $(x, x, e) = 0$ for $x \in R_{1/2}$, whence linearization implies $(x_{ij}, y_{ji}, e) + (y_{ji}, x_{ij}, e) = 0$. Thus $(x_{ij}, y_{ji}, e) = -(y_{ji}, x_{ij}, e) \in R_{ji} \cap R_{ij} = 0$ by Lemma 9. Since obviously $(x_{ij}, e, y_{ji}) = 0$, this establishes

(xvi) $(x_{ij}, y_{ji}, e) + (x_{ij}, e, y_{ji}) = 0$ for $i \neq j$.

In particular, we note that $(x_{ij}, y_{ji}, e) = 0 = (e, x_{ij}, y_{ji})$ and Lemma 9 now imply $x_{ij}y_{ji} \in R_{ii}$.

Next let $u = (x_{ij}, y_{ij}, e) + (x_{ij}, e, y_{ij})$. We shall show $u \in B$. It may be verified from Lemma 9 that $u \in R_{ij}$, so $z_{ij}u \in R_{ij} + R_{ji}$. Also, linearized (2), (xvi) and Lemma 9 imply $z_{ji}u = z_{ji}[(x_{ij}, y_{ij}, e) + (x_{ij}, e, y_{ij})] = [(z_{ji}x_{ij}, y_{ij}, e) + (z_{ji}x_{ij}, e, y_{ij})] - [(z_{ji}, y_{ij}, e) + (z_{ji}, e, y_{ij})]x_{ij} = 0$, since as noted $z_{ji}x_{ij} \in R_{jj}$. Finally, going to the opposite ring and using $e \in N_F(R)$, we have $uz_{ij} \in R_{ij} + R_{ji}$ and $uz_{ji} = 0$ as well. Since $B = 0$, this shows $(x_{ij}, y_{ij}, e) + (x_{ij}, e, y_{ij}) = 0$ for $i \neq j$ and thereby completes the proof of the theorem.

Now in [4] Rich showed that if a ring R is prime with idempotent $e \neq 1$ and if every idempotent of R lies in $N_A(R)$, then R must be alternative. Hence Theorem 5 leads directly to our main result.

THEOREM 6. *Let R be a prime derivation alternator ring with idempotent $e \neq 1$ and characteristic $\neq 2$. If R is without nonzero nil ideals of index 2, then R is alternative.*

COROLLARY. *A simple derivation alternator ring with idempotent $e \neq 1$ and characteristic $\neq 2$ is alternative.*

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